І. ПРОБЛЕМИ МЕТОДИКИ НАВЧАННЯ МАТЕМАТИЧНИХ ДИСЦИПЛІН

SOLVING COMPETITIVE PROBLEMS IN THE NUMBER THEORY WITH THE VIEW OF IMPROVEMENT OF MATHEMATICAL TRAINING OF STUDENTS

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У статті висвітлено аспекти теоретико-числових задач, що сприяють розвитку логічного мислення, поглибленню знань з математики і можуть бути використані для підготовки учнів до олімпіад та позакласної роботи.

The aspects of theoretic-numerical problems that contribute to the development of logical thinking, deepening knowledge of mathematics and can be used to prepare students for competitions and extracurricular activities are examined in this article.

The problem. The democratization of the educational system of Ukraine requires mathematical science to find new methodological technologies that would have provided along with a high level of theoretical and practical training in mathematics also refocus of the educational process on the student's personality, favorable conditions for achieving the chosen level of knowledge. In today's schools one of the most effective ways to motivate students to study, to the cognitive activity, development of their creative abilities, to deepen and broaden students' knowledge are the subject school contests that promote the development of skills to solve problems of increased complexity, a defense of students' research works in the SAS (Small Academy of Sciences). The topic of this article was chosen due to the fact that the number-theoretic problems are frequently encountered in mathematical contests at various levels, entrance exams, but still are difficult for students and teachers.

Analysis of previous research. The structure and content of mathematical study, continuity in learning and teaching mathematics and professional orientation investigated M.I. Burda, V.G. Bevz, N.A. Tarasenkova, V.O. Shvets, T.M. Hmara, N.M. Voynalovych and others; forming a creative individual of the student, development of creative thinking while studing mathematics – Z.I. Slyepkan, O.S. Chashechnykova, O.I. Skafa, V.A. Kushnir, R.J. Rizhnyak, L.I. Lutchenko and others. Systematic approach to organizing and solving unusual problems investigated V.I. Michailovsky, I.M. Mitelman, O.G. Ganyushkin, V.V. Plakhotnik, M.V. Pratsovyty, O.M. Vorony, I.V. Fedak, V.M. Radchenko, M.O. Perestyuk, M.S. Dobosevych, V.A. Yasinsky, V.N. Nagorny, V.O. Borisova, V.M. Leyfura, V.S. Mazorchuk, V.A. Vyshens'kyi, M.V. Kartashov, K.V. Rabets, O.J. Teplinsky, V.V. Nekrashevich, O.O. Kurchenko, N.M. Shunda, G.V. Apostolova and others. Research of the theoretic-numeral component in the system of mathematical education can be found in the works by V.O. Shvets, V.A. Yasinsky, V.V. Yasinsky, V.V. Plakhotnik and others.

Goals of the article. Level of mathematical schooling of the student is characterized primarily by his ability to solve problems. It is no coincidence that the current practice of teaching mathematics much of the training time is devoted to solving the problem.

The primary means of mathematical thinking is solving problems. Obviously, we don't mean training exercises, but the unusual tasks, which solution, as either non-standard solutions of traditional problems, as an important component in the development of creative abilities of the individual. Problems motivate students to nominate and justificate certain assumptions, construct fragmentary theoretical generalizations, contributing in this way the formation of students' creative, heuristic thinking and commitment to research. In this regard a significant role in the mathematical training of the student is to be given to solving problems.

The purpose of this article is to enlighten methodical aspects of solution the various types of competitive theoretical and numerical problems that are relevant at this time and that can be measured in terms of circle work with students of 10.-11. physical and mathematical classes of

Pedagogical Lyceum and more than a decade long experience of work with students from the group "Mathematics" of Kirovohrad regional office of the Small Academy of Sciences.

Use of theoretical and numerical problems for improving mathematics scholarship of the students. Solving unusual problems in the classroom, circles and other types of extracurricular activities allows students to gain experience in comparison, observation, identifying simple mathematical regularities, puting forward hypotheses that need to be proofed. Thus, the conditions for the development of deductive reasoning arise. In addition, these tasks can help teachers in the education of moral personality traits as assiduity, persistence in achieving goals, perseverance, diligence and so on. Finally, on efficiency of problems use in teaching mathematics largely depends not only the quality of training, education and development of students, but also the level of their practical qualification to the future activities in any area of the economy and culture.

A significant part of theoretical and numerical problems are problems to find a rational (integer, positive) solutions of equations and their systems. Diophantine equation is the equation with integer coefficients of any number of variables and whatever degree. And there are integer or rational solutions, and the number of variables in the Diophantine equation is greater than the number of equations. No contest, mathematical competition passes without Diophantine equation or problem which leads to solving such equations. In the literature we can find a description of the methods of solving Diophantine equations: localization and enumeration methods, graphical method, method of factoring and the method of descent. In our opinion, it is also appropriate to use the theory of divisibility to solve these equations.

Let us consider the methodological aspects of the solution of the Diophantine equation using method of factorization on these examples.

Problem #1. Find all solutions of the equation $x^2 - y^2 = 3$.

Solving. $(x-y)(x+y) = 3$, and since x, y are integers, their sum and difference are integers either, so we have a set of four systems: $\begin{cases} x-y=1 \\ x+y=3 \end{cases}$, $\begin{cases} x-y=3 \\ x+y=1 \end{cases}$, $\begin{cases} x-y=-1 \\ x+y=-3 \end{cases}$, $\begin{cases} x-y=-3 \\ x+y=-1 \end{cases}$. There is no other options, because the number 3 is simple.

Solution: $(2;1);(2;-1);(-2;-1);(-2;1)$.

While solving a problem like this for the first time it is advisable to record all the possible options and solve them. This helps to accustom students to mathematical tidiness while solving problems, precise thinking, to exclude unnecessary haste, etc. However, after having solved enough problem this kind it is desirable to investigate the form of equation (symmetry to Ox, Oy, O) and then find all solutions of sufficient only for non-negative. Taking into account the ratio between the difference and the sum of non-negative integers $x - y \le x + y$, there is only one option $\begin{cases} x-y=1 \\ x+y=3 \end{cases}$, where we get a couple $(2;1)$, and then all the solutions differ only by signs.

Solution: $(\pm 2; \pm 1); (\pm 2; \mp 1)$.

Conclusions obtained for Problem 1, can be used for the next exercise.

Problem #2. Find all solutions of the equation $x^2 - y^2 = 12$.

Solving. Similar to Problem 1, we will look for solutions in the first quarter. With three possibilities for the number 12: $12 = 1 \cdot 12$, $12 = 2 \cdot 6$, $12 = 3 \cdot 4$, we get only one integer solution $(4;2)$ for $\begin{cases} x-y=2 \\ x+y=6 \end{cases}$.

Solution: $(\pm 4; \pm 2); (\pm 4; \mp 2)$

In our opinion, while the first solution of such problem it is advisable to prescribe all systems of the complex and find all solutions (including fractions). But after a while it is worth examining why some of the systems could not have integer solutions (as the numbers $x - y, x + y$ are of equal parity, because their sum is equal to $2x$, they can not be one even, and another odd at the same time). The work should be carried out with fixing the most troublesome episodes of the solution.

After these considerations it is useful to prove the following statement.

Problem #3. Prove that the equation has no integer solutions: $x^2 - y^2 = 62$.

Proof. The right side is the number is even, and therefore the left is even either, that is why some of the factors is divisible by two (theoretical framework: if the product of integers is divisible by a prime number, then at least one of the factors is divided by this number). Multipliers of the left part are of the same parity, so both are even. Then the left side is divisible by four, and the right isn't. Impossible.

It is to be mentioned that one and the same themes, that are dealt with in different forms, should have different problems complex and sometimes different ways to solve them.

Problem #4. Find all the pairs of positive integers (x, y) that satisfy the equation $x^2y = x^2 + 48y$.

Solving. Let's schedule $x^2y = x^2 + 48y$ for factors: $(x^2 - 48)(y-1) = 48$, and then $48 = 2^4 \cdot 3$, $\tau(48) = 5 \cdot 2 = 10$, factors of the number 48: $\{1,2,3,4,6,8,12,16,24,48\}$. We have a set of 10 systems that allow only two integer solutions $(7,49)$; $(8,4)$.

Another method for solving Diophantine equations is the method of localization and enumeration. However, while solving this equations this way you can use the properties of divisibility by 2, 3, 5 and others.

Problem #5. Find natural solutions to the equation $2x^3 + 3xy = 728$.

Solving. Since $2x^3 < 728$, then $x < \sqrt[3]{364}, x < 8$, consequently $x \in \{1, 2, \ldots, 7\}$ (localization). Since the left side is divisible by x equal, then if $x \in \{3, 5, 6\}$ the left side is divisible by $3, 5, 3$ respectively, so the right side isn't divisible by three or five (therefore, it's impossible). So only $x \in \{1, 2, 4, 7\}$ are to take over, and we have the answer $(1;242)$; $(4;50)$; $(7;2)$.

Need to solve a Diophantine equation erases either while solving systems of Diophantine equations.

Problem #6. To find all the integer solutions to the system: $\begin{cases} 12x - 10y + 7z = 2 \\ 4x - 3y + 2z = 1 \end{cases}$.

Solving. Let's exclude variable x from the system of equations, we obtain an equation that depends on two variables (y, z). Let us solve the Diophantine equation relative to y, z and substitute the obtained values in x:

$$\begin{cases} 12x - 10y + 7z = 2 \\ 4x - 3y + 2z = 1 \quad |\cdot(-3) \end{cases} \Rightarrow \begin{cases} 12x - 10y + 7z = 2 \\ 4x - 3y + 2z = 1 \quad |\cdot(-3) \end{cases} \Rightarrow \quad \Rightarrow \begin{cases} y = z + 1 \\ x = \dfrac{3y - 2z + 1}{4} \end{cases}.$$

Then $x = \dfrac{3(z+1) - 2z + 1}{4}$, $x = \dfrac{z}{4} + 1$. Since x is integer, z must be divisible by 4: $z = 4t$; $y = 1 + 4t$; $x = 1 + t$, $t \in Z$.

Verification:

$$\begin{cases} 12(1+t) - 10(1+4t) + 7 \cdot 4t = 2 \\ 4(1+t) - 3(1+4t) + 2 \cdot 4t = 1 \end{cases} \Leftrightarrow \begin{cases} 12 + \underline{12t} - 10 - \underline{40t} + \underline{28t} = 2 \\ 4 + \underline{4t} - 3 - \underline{12t} + \underline{8t} = 1 \end{cases} \Leftrightarrow \begin{cases} 2 = 2 \\ 1 = 1 \end{cases}.$$

Solution: $x = 1 + t, y = 1 + 4t, z = 4t, t \in Z$.

It is reasonable to accustom children to self-control. In our opinion the solutions to the problems in textbooks should not soothe. It is advisable to refer to them (if they are given) only after verification. An important means of controlling the accuracy of the results is re-execution problem by another means.

Problem #7. On the number line all the numbers that when divided by 84 give the remainder 35 are marked yellow, and blue – all the numbers that when divided by 56 give the remainder 3. Find the shortest distance between the yellow and blue dot.

Solving. We have the numbers $84x+35$ and $56y+3$ where $x,y\in Z$. The distance between them is $d=\left|(84x+35)-(56y+3)\right|=\left|84x-56y+32\right|$. Since $x,y\in Z$, the expression $(84x-56y+32)\vdots 4$, ie $d\vdots 4$. Let's verify whether a distance can be equal to four. It is enough to show that there are solutions of at least one of two equations: $84x-56y+32=-4$ or $84x-56y+32=4$. The first equation has no solutions: $84x-56y=-36\Leftrightarrow 3x-2y=-9/7$. The second equation $84x-56y=-28\Leftrightarrow 3x-2y=-1$ has integer solutions: $x=1+2t, y=2+3t$, where $t\in Z$. And so the minimum possible distance between these points is equal to 4. For example, between blue 115 and yellow dot 119.

Much of the theoretical-numeric problems of the contest level are problems in which the simplicity of numbers is to explore or all the prime numbers that satisfy certain conditions are to find. Unfortunately, the literature doesn't describe the mechanism of solution, which feature it is appropriate to in a particular case and why. We encourage our students to use parity (the only prime even number is 2), divisibility by three (explore the remainder of the division by three), divisibility by 10 (by which number may a prime number end), and to do it consistently, depending on if results are positive or not.

Problem #8. Find the smallest positive prime number p for which $n=p^2+3p+11$ is a prime. Find all positive primes of this type.

Solving. Substituting consistently positive primes $p=2, p=3$ we obtain: at $p=2,\ n=21$ it is not a prime, and if $p=3$ obtain $n=29$ – a prime. Since $n=p^2+3p+11=(p^2-1)+3(p+4)$ is divisible by three and isn't equal to three, therefore it is not prime at all primes $p>3$ (theoretical framework: by the prime $p>3$ $(p^2-1)\vdots 3$).

Solution: when $p=3$, $n=29$.

Problem #9. It is known that the numbers $p,q,q+3,7p+5q,p-5q$ are positive, prime, pairwise distinct integers. Find p and q.

Solving. Let's use divisibility by two. The second and third numbers $q,q+3$ are prime numbers of different parity. And so one of them is equal to two. Since the numbers are positive, $q=2, q+3=5$. The fourth and fifth numbers are primes with the same parity (odd), the result is not obtained, let's proceed to use divisibility by three. The numbers that are left: $p, 7p+10=(6p+9)+(p+1), p-10=(p-1)-9$ give different remainders when divided by three. And so one of the numbers is equal to three according to the condition of primality. By $p=3$ the last number isn't positive; by $7p+10=3$ the number p is not natural, therefore we have a single possibility: $p-10=3$. And then $p=13;\ 7p+10=101$ (a prime, because it is not divisible by 2, 3, 5, 7).

Solution: $p=13,\ q=2$.

Problem #10. Find all triples of primes $(p;\ q;\ r)$ for which equality $pqr=p^2+q^2+r^2$ is true.

Solving. Use the divisibility by two, can one of the numbers be even? Let us suppose that $p=2$. Then $2qr=4+q^2+r^2\Leftrightarrow (q-r)^2=-4$. Impossible. Let's proceed to use divisibility by three. We suppose that the numbers p,q,r are not divisible by three, then p,q,r have a look $(3k\pm 1)$, and then p^2,q^2,r^2 have the form $(3k\pm 1)^2\equiv 1(\mathrm{mod}\,3)$; the left part of the equality is not divisible by three, and the right side when divided by three gives a zero remainder: that is divisible by three. Impossible. Therefore, one of the numbers (eg, p) is divisible by three (and is a prime), so $p=3$. So $p=3$, then $3qr=9+q^2+r^2$. Let's suppose that the numbers q,r are not divisible by three, the same: q^2,r^2 have a look $(3k\pm1)^2\equiv1(\mathrm{mod}3)$; but now the left part of the equality is divisible by three, and the right side by dividing by three gives the remainder two: that is not divisible by three. Impossible. Therefore, one of the numbers (eg, q) is divisible by three, is a prime, so $q=3$. And then $r \vdots 3 \Rightarrow r=3$.

Solution: one set $(3,3,3)$.

Problem #11. Find all natural values n for which three numbers $n^2+3n-8,\ 4n^2-n+3,$ n^2+6n-5 are positive primes.

Solving. Obviously, the problem cannot be solved by means of localization. Let's investigate the unknown numbers. Since among the primes there is only one even number (number two), and all the others are odd, given that the sum of the three numbers $(n^2+3n-8)+(4n^2-n+3)+(n^2+6n-5)=6n^2+8n-10$ is an even number, we conclude that all of them cannot be odd. Thus, at least one of them is even, and therefore equal to two. If it is the first number, we obtain $n^2+3n-8=2 \Leftrightarrow (n-2)(n+5)=0$, and then $n=2$ and the desired numbers are $2;17;11$ respectively. The second number isn't equal to two, and if the third number is two, then $n=1$ and the first number is no longer natural.

Solution: $n=2$ (numbers $2;17;11$).

Problem #12. Find all the prime p that can be represented as $p=a^4+b^4+c^4-3$ where a,b,c are some (not necessarily distinct) primes.

Solving. If all the numbers a,b,c are odd primes, then the number p is even (more than 2), so is not prime. So some of the numbers a,b,c is even simple, that is equal to two. Let's suppose that $c=2$. Then $p=a^4+b^4+13$. Let's investigate all the possibilities: a,b are both even, of different parity, both odd: if a,b are both even (prime), then $a=b=2$ and $p=45$ is not a prime; if a and b are of different parity, then the number p is even greater than two, so not a prime; then a,b are both odd. Let's use divisibility by two and divisibility by three. All the numbers look like $6k,\ 6k\pm1,\ 6k\pm2, 6k+3$. As the number a and b are odd, they cannot have the form $6k,\ 6k\pm2$. So the following possibilities: $6k\pm1, 6k+3$. Let's suppose that $a=6k\pm1, b=6m\pm1$. Then $p=(6k\pm1)^4+(6m\pm1)^4+13\equiv1+1+1\equiv0(\mathrm{mod}3)$ is divisible by three or more than three, so is not prime. If one of the numbers, for example a, has the form $a=6k+3$, then since the number a is prime $a=3$, so $p=b^4+94$.

Since the number b is odd, it may end with the number 1, 3, 5, 7, 9, and then b^4 ends, respectively, with 1, 1, 5, 1, 1, and the number $p=b^4+94$, respectively, with 5, 5, 9, 5, 5. The first two and the last two cases correspond to non-prime p multiple of five. The middle case shows that an odd prime number b ends with 5, so $b=5, p=5^4+94=719$. Since 719 is not divisible by any prime number 2, 3, 5, 7, 11, 13, 17, 19, 23, that doesn't exceed $\sqrt{719}$, it is prime. Thus, the numbers a,b,c can only be $2,3,5$ (any order), and number $p=719$.

Solution: $p=719$.

Mathematics, being an exact science, can cultivate critical thinking skills since first grade education. The school of the second and third stages create special opportunities for this process. The basic form of this is the precise fixing of the guidelines, written form of all the calculations and assertions and checking the results. Use of analogies must be justified, we must prove the acceptability of this analogy not so much with resemblance as with common causes. It is important to bring up the courage to formulate hypotheses. At appropriate stages of the lesson the situation of "brainstorming" is desirable. The creative personality doesn't only prove or refute certain statements, but constructs, "guesses" the new ones.

Conclusions. Ever expanding range of elective classes, opportunity to study in a circle, at the extramural physical-mathematical school (ZFMSH), SAS helps student focus on the problems that he has chosen for his own research, which will contribute to the full and harmonious development of personality. It should be remembered, because today's students will have to deal with problems that are not yet resolved, acquire specialties that do not yet exist, use technologies that have not yet been created.

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