Axiomatics for Multivalued Dependencies in Table Databases: Correctness, Completeness, Completeness Criteria

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Abstract. Axiomatics for multivalued dependencies in table databases and axiomatics for functional and multivalued dependencies are reviewed; the completeness of these axiomatics is established in terms of coincidence of syntactic and semantic consequence relations; the completeness criteria for these axiomatic systems are formulated in terms of cardinalities (1) of the universal domain D, which is considering in interpretations, and (2) the scheme R, which is a parameter of all constructions, because only the tables which attributes belong to this scheme R are considering.

The results obtained in this paper and developed mathematical technique can be used for algorithmic support of normalization in table databases.

Keywords: table databases, functional dependencies, multivalued dependencies, completeness of axiomatic system.

1 Introduction

Data integrity of relational (table) databases is dependent on their logical design. To eliminate the known anomalies (update, insertion, deletion) the database normalization is required. Analysis of the current state of normalization theory in relation databases indicates that the accumulated theoretical researches not enough to satisfy the needs of database developers (see, survey [1] based on 54 sources). This is evidenced works devoted to the ways of solving existing problems of designing database schemas (see, for example, [2]) and improvement of algorithmic systems for normalization (see, for example, [3]).

The process of normalization is based, in particular, upon functional and multivalued dependencies theory the foundation of which is made by corresponding axiomatics and their completeness. The overview of research sources has shown that these axiomatic systems lack the proof of completeness that would comply with mathematical rigor; in fact, these vitally important results have a seemingly announcement status.

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Known CASE-tools (Computer-Aided Software Engineering tools) (for example, ERwin[1], Vantage Team Builder (Cadre), Silverrun[2]) perform normalization to 3NF (Third Normal Form). The results obtained in this paper can be used for development of the CASE-tools which support normalization. Since without proper mathematical results for the correctness and completeness of axiomatics the algorithms don't have foundation and keep heuristic in nature.

Features of this paper is clear separation of syntactic and semantic aspects.

All undefined concepts and notation are used in understanding of monograph [4], in particular, $s \mid X$ – restriction the row s to the set of attributes X. Symbol □ means the end of statement or proof, symbol ▫ – end of logical part of proof.

2 Axiomatic for Multivalued Dependencies

Let t – a table, *R* – the scheme of the table *t* (finite set of attributes); *X, Y, W, Z* – subsets of scheme *R*; s, s_1, s_2 – the rows of table t. Henceforth we shall assume that set R and *universal domain* D *(*the set, from witch attributes take on values in interpretations*)* are fixed.

A multivalued dependence (MVD) $X \to\to Y$ is valid on the table *t* of the scheme *R* (see, for example, [4]), if for two arbitrary rows s_1, s_2 of table *t* which coincide on the set of attributes *X*, exists row $s_3 \in t$ which is equal to the union of restrictions of the rows s_1, s_2 to the sets of attributes $X \cup Y$ and $R \setminus (X \cup Y)$ respectively:

$$(X \to\to Y)(t) = true \overset{def}{\Leftrightarrow} \forall s_1, s_2 \in t (s_1 \mid X = s_2 \mid X \Rightarrow$$

$$\Rightarrow \exists s_3 \in t (s_3 = s_1 \mid (X \cup Y) \cup s_2 \mid R \setminus (X \cup Y))) .$$

Thus, from the semantic point of view MVD – parametric predicate on tables of the scheme *R* defined by two (finite) parameter-sets of attributes X, Y.

Structure of table t, which complies with MVD $X \to\to Y$, can be represented using the following relation. We say that rows s_1, s_2 of table t are in the relation $=_X$, if they coincide on the set of attributes *X*:

$$s_1 =_X s_2 \overset{def}{\Leftrightarrow} s_1 \mid X = s_2 \mid X .$$

It is obvious that relation $=_X$ is equivalence relation and therefore it partitions the table s into equivalence classes, which are as follows:

$$[s]_{=_X} = \{s \mid X\} \otimes \pi_Y([s]_{=_X}) \otimes \pi_{R \setminus (X \cup Y)}([s]_{=_X}) ,$$

where s – arbitrary representative of the class.

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[1] http://erwin.com/products/detail/ca_erwin_process_modeler/

[2] http://www.silverrun.com/

A table $t(R)$ is the model of a set of MVDs G, if each MVD $X \to\to Y \in G$ is valid on table $t(R)$:

$$t(R) \text{ is the } \textit{model} \text{ of } G \overset{def}{\Leftrightarrow} \forall (X \to\to Y)(X \to\to Y \in G \Rightarrow (X \to\to Y)(t) = true).$$

Here and in the sequel the notion $t(R)$ stand for the table t of the scheme R.

The next axioms and inference rules are valid [5].

Axiom of reflexivity: $\forall t(X \to\to Y)(t) = true$, where $Y \subseteq X$.

Axiom: $\forall t(X \to\to Y)(t) = true$, where $X \bigcup Y = R$.

Rule of *complementation*: $(X \to\to Y)(t) = true \Rightarrow (X \to\to R \setminus (X \bigcup Y))(t) = true$.

Rule of *augmentation*:
$(X \to\to Y)(t) = true \,\&\, Z \subseteq W \Rightarrow (X \bigcup W \to\to Y \bigcup Z)(t) = true$.

Rule of *transitivity:*
$(X \to\to Y)(t) = true \,\&\, (Y \to\to Z)(t) = true \Rightarrow (X \to\to Z \setminus Y)(t) = true$.

As an example we give the proof of the axiom of reflexivity.

Proof. Let s_1 and s_2 be the rows of table t for which $s_1 | X = s_2 | X$ is carried out. We show that the row $s_1 | (X \bigcup Y) \bigcup s_2 | R \setminus (X \bigcup Y)$ belongs to the table t. Restrict both parts of equality $s_1 | X = s_2 | X$ to the set Y: $(s_1 | X) | Y = (s_2 | X) | Y$. According to the property of restriction operator ($(U|Y)|Z = U|(Y \bigcap Z)$) [1, p. 24]) it follows $s_1|(X \bigcap Y) = s_2|(X \bigcap Y)$. Consequently, and from the condition $Y \subseteq X$ we get $s_1|Y = s_2|Y$. According to the distributive property of restriction operator $(U | \bigcup_i X_i = \bigcup_i (U | X_i))$ [1, p. 24] it follows: $s_1 | (X \bigcup Y) \bigcup s_2 | R \setminus (X \bigcup Y) =$
$= s_1 | X \bigcup s_1 | Y \bigcup s_2 | R \setminus (X \bigcup Y) = s_2 | X \bigcup s_2 | Y \bigcup s_2 | R \setminus (X \bigcup Y) = s_2 | (X \bigcup Y \bigcup$
$\bigcup R \setminus (X \bigcup Y)) = s_2 | R = s_2$. □

The proof of other axiom and rules is given similarly.

A MVD $X \to\to Y$ is semantically deduced from the set of MVD's G, if at each table $t(R)$, which is the model of set G, MVD $X \to\to Y$ is valid too:

$$G \models X \to\to Y \overset{def}{\Leftrightarrow} \forall t(R)(t \text{ is the model of the } G \Rightarrow (X \to\to Y)(t) = true).$$

From above-mentioned axioms and inference rules follow corollaries.

Lemma 1. The next properties of the semantic consequence relation are valid:

1) $\varnothing \models X \to\to Y$ for $Y \subseteq X$;

2) $\varnothing \models X \to\to Y$ for $X \bigcup Y = R$;

3) $G \models X \to\to Y \Rightarrow G \models X \to\to R \setminus (X \bigcup Y)$;

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4) $G \models X \to\to Y \ \& Z \subseteq W \Rightarrow G \models X \bigcup W \to\to Y \bigcup Z$;

5) $G \models X \to\to Y \ \& \ G \models Y \to\to Z \Rightarrow G \models X \to\to Z \setminus Y$;

6) $G \models X \to\to Y \ \& \ G \models Y \to\to Z \ \& Z \bigcap Y = \varnothing \Rightarrow G \models X \to\to Z$. □

A MVD $X \to\to Y$ is syntactically derived from the set of MVD's G with respect to the scheme R ($G \mid -_R X \to\to Y$), if there is a finite sequence of MVD's $\varphi_1, \varphi_2, ..., \varphi_{m-1}, \varphi_m$ where $\varphi_m = X \to\to Y$ and for all $\forall i = \overline{1, m-1}$ each φ_i is either the axiom of reflexive or belongs to G, or is derived with some inference rule for MVD's (complementation, augmentation, transitivity) from the previous in this sequence φ_j, φ_k, $j, k < i$.

Let sequence $\varphi_1, \varphi_2, ..., \varphi_{m-1}, \varphi_m$ be called proof, following the tradition of mathematical logic [6].

Let there be given certain set of MVD's G. *Closure* $[G]_R$ is a set of all MVD's, that are syntactically derived from G:

$$[G]_R \stackrel{def}{=} \{X \to\to Y \mid G \mid -_R X \to\to Y\}.$$

For notational convenience, we write $\mid -$ for $\mid -_R$.

Lemma 2. Next properties are valid:

1) $G \subseteq [G]$ (*increase*);

2) $[[G]] = [G]$ (*idempotency*);

3) $G \subseteq H \Rightarrow [G] \subseteq [H]$ (*monotonicity*). □

Proof. Let's prove proposition 1. Let MVD's $X \to\to Y \in G$, then $G \mid - X \to\to Y$ with one number of steps of proving, hence, $X \to\to Y \in [G]$. ▫

Let's prove proposition 2. According to property 1, we have $[G] \subseteq [[G]]$. Let us prove the reverse inclusion $[[G]] \subseteq [G]$. Let $X \to\to Y$ – arbitrary MVD, such that $X \to\to Y \in [[G]]$. Then there is a finite sequence MVD's $\varphi_1, \varphi_2, ..., \varphi_{m-1}, \varphi_m$, such that $\varphi_m = X \to\to Y$ and for all $\forall i = \overline{1, m-1}$ each φ_i is either the axiom of reflexive property or belongs to $[G]$, or is derived with the help of some inference rule for MVD's from the previous in this sequence φ_j, φ_k, $j, k < i$. Let us make a new sequence according to such rules:

- if φ_i is the axiom of reflexivity, then we write down this MVD without any changes;
- if $\varphi_i \in [G]$, then according to the definition of closure this MVD has a finite proof $\psi_1, ..., \psi_{l-1}, \psi_l$ from G. Instead of MVD φ_i let's insert this proof;
- if φ_i is derived according to any inference rule from the previous in this sequence MVD's φ_j, φ_k, $j, k < i$, then also we write down φ_i without any changes.

Clearly, created in such a way the sequence is a proof of MVD $X \to\to Y$ from G that is $G \mid - X \to\to Y$, hence, $X \to\to Y \in [G]$. ▫

Thereby, operator $G \mapsto [G]$ is closure operator in terms of [7].

Observe that properties of operator $G \mapsto [G]$ listed in Lemma 2, are carried out in axiomatic systems (see, for example, [8]).

From reflexivity axiom and inference rules indicated above is possible to get other inference rules for MVD's [2, 6].

Rule of *pseudo-transitivity*:

$$\{X \to\to Y, Y \cup W \to\to Z\} \mid - X \cup W \to\to Z \setminus (Y \cup W).$$

Rules of *difference*:

a) $\{X \to\to Y\} \mid - X \to\to Y \setminus X$;

b) $\{X \to\to Y \setminus X\} \mid - X \to\to Y$;

c) $\{X \to\to Y\} \mid - X \to\to R \setminus Y$.

Rule of union: $\{X \to\to Y_1, X \to\to Y_2\} \mid - X \to\to Y_1 \cup Y_2$.

Rules of decomposition:

a) $\{X \to\to Y_1, X \to\to Y_2\} \mid - X \to\to Y_1 \cap Y_2$;

b) $\{X \to\to Y_1, X \to\to Y_2\} \mid - X \to\to Y_1 \setminus Y_2$. □

Lemma 3. The next properties are valid for $n = 2, 3, \ldots$:

1) $\{X \to\to Y_1, \ldots, X \to\to Y_n\} \mid - X \to\to Y_1 \cup \ldots \cup Y_n$;

2) $\{X \to\to Y_1, \ldots, X \to\to Y_n\} \mid - X \to\to Y_1 \cap \ldots \cap Y_n$. □

The proofs of this lemma constructed by the induction in the n, according the rules of augmentation and transitivity.

3 Axiomatic for Multivalued Dependencies and Functional Dependencies

It will be recalled that a functional dependence $X \to Y$ is valid on the table *t,* if for two arbitrary rows s_1, s_2 of table *t* which coincide on the set of attributes *X*, their equality on the set of attributes *Y* is fulfilled (see, for example [4]), that is:

$$(X \to Y)(t) = true \overset{def}{\Leftrightarrow} \forall s_1, s_2 \in t\left(s_1|X = s_2|X \Rightarrow s_1|Y = s_2|Y\right).$$

Let there be given a sets F and G of FD's and MVD's respectively. A table $t(R)$ is the model of a set $F \cup G$, if each dependency $\varphi \in F \cup G$ is valid at table t:

$$t(R) \text{ is } \textit{model} \text{ of } F \cup G \overset{def}{\Leftrightarrow} \forall \varphi\ (\varphi \in F \cup G \Rightarrow \varphi(t) = true).$$

Mixed inference rules for FD's and MVD's are valid [5].

1. Rule of *extension* FD to MVD
   $(X \to Y)(t) = true \Rightarrow (X \to\to Y)(t) = true$.
2. $(X \to\to Z)(t) = true$ & $(Y \to Z')(t) = true$ & $Z' \subseteq Z$ & $Y \bigcap Z = \varnothing \Rightarrow$
   $\Rightarrow (X \to Z')(t) = true$.

The proof of *extension* rule is presented, for example, in the monograph [1, p. 73]). Let's prove mixed inference rules for FD's and MVD's (proposition 2).

Proof. Let s_1 and s_2 be the rows of table t for which $s_1 \mid X = s_2 \mid X$ is carried out and MVD $X \to\to Z$ holds for table t. Therefore, there is row $s_3 \in t$ that $s_3 = s_1 \mid X \bigcup s_1 \mid Z \bigcup s_2 \mid R \setminus (X \bigcup Z)$. Let FD $Y \to Z'$ holds on table t, where $Z' \subseteq Z$ and $Y \bigcap Z = \varnothing$.

First we show that equality $s_2 \mid Y = s_3 \mid Y$ for rows s_2 i s_3 is fulfilled. From equalities $s_2 \mid X = s_3 \mid X$ (by assumption, $s_1 \mid X = s_2 \mid X$ and by construction of row s_3, $s_1 \mid X = s_3 \mid X$) and $s_2 \mid R \setminus (X \bigcup Z) = s_3 \mid R \setminus (X \bigcup Z)$ (by construction of row s_3) we have $s_2 \mid (R \setminus (X \bigcup Z) \bigcup X) = s_3 \mid (R \setminus (X \bigcup Z) \bigcup X)$, that is [3] $s_2 \mid (R \setminus Z \bigcup X) = \; = s_3 \mid (R \setminus Z \bigcup X)$; hence and from inclusion $R \setminus Z \subseteq R \setminus Z \bigcup X$ we have equality $s_2 \mid (R \setminus Z) = s_3 \mid (R \setminus Z)$. By condition, we have $Y \bigcap Z = \varnothing$ therefore $Y \subseteq R \setminus Z$, hence, $s_2 \mid Y = s_3 \mid Y$.

By condition, we have $(Y \to Z')(t) = true$, hence $s_2 \mid Z' = s_3 \mid Z'$. Since $s_1 \mid Z = s_3 \mid Z$ (by construction of row s_3), then from inclusion $Z' \subseteq Z$ it follows $s_1 \mid Z' = s_3 \mid Z'$. Thus, for rows s_1 and s_2 which coincide on the set of attributes X , equality $s_1 \mid Z' = s_2 \mid Z'$ is fulfilled. Thus, FD $X \to Z'$ holds for table t . □

FD or MVD φ is semantically deduced from the set of dependencies $F \bigcup G$, if at each table $t(R)$, which is the model of a set of dependencies $F \bigcup G$, dependency φ is valid too:

$$F \bigcup G \models \varphi \overset{def}{\Leftrightarrow} \forall t(R)(t(R) - \text{model of } F \bigcup G \Rightarrow \varphi(t) = true) .$$

From above-mentioned mixed inference rules for FD's and MVD's follow corollaries (the properties of semantic consequence relation):

1. $F \models X \to Y \Rightarrow F \models X \to\to Y$;
2. $G \models X \to\to Z$ & $F \models Y \to Z'$ & $Z' \subseteq Z$ & $Y \bigcap Z = \varnothing \Rightarrow F \bigcup G \models X \to Z'$.

[3] It is required to take into account the succession of set-theoretic equalities $R \setminus (X \bigcup Z) \bigcup Z = (R \setminus X \bigcap R \setminus Z) \bigcup Z = (R \setminus X \bigcup Z) \bigcap \; (R \setminus X \bigcup Z) \bigcap \; (R \setminus Z \bigcup Z) =$ $= (R \setminus X \bigcup Z) \bigcap R = R \setminus X \bigcup Z$.

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Lemma 4. Let H_1 i H_2 – the sets of dependencies (FD's or MVD's) and T_1, T_2 – the sets of all their models respectively. Then implication $H_1 \subseteq H_2 \Rightarrow T_1 \supseteq T_2$ is carried out. □

Corollary 1. The next properties of the semantic consequence relation are valid:
1) $F \models \varphi \Rightarrow F \bigcup G \models \varphi$;
2) $G \models \varphi \Rightarrow F \bigcup G \models \varphi$. □

Lemma 5. The next properties of semantic consequence relation are valid:
1) $F \models X \to Y \Rightarrow F \bigcup G \models X \bigcup Z \to Y \bigcup Z$ for $Z \subseteq R$;
$F \bigcup G \models X \to Y \Rightarrow F \bigcup G \models X \bigcup Z \to Y \bigcup Z$ for $Z \subseteq R$;
2) $F \models X \to Y \;\&\; F \models Y \to Z \Rightarrow F \bigcup G \models X \to Z$;
$F \bigcup G \models X \to Y \;\&\; F \bigcup G \models Y \to Z \Rightarrow F \bigcup G \models X \to Z$;
3) $G \models X \to\to Y \Rightarrow F \bigcup G \models X \to\to R \setminus (X \bigcup Y)$;
$F \bigcup G \models X \to\to Y \Rightarrow F \bigcup G \models X \to\to R \setminus (X \bigcup Y)$;
4) $G \models X \to\to Y \;\&\; Z \subseteq W \Rightarrow F \bigcup G \models X \bigcup W \to\to Y \bigcup Z$;
$F \bigcup G \models X \to\to Y \;\&\; Z \subseteq W \Rightarrow F \bigcup G \models X \bigcup W \to\to Y \bigcup Z$;
5) $G \models X \to\to Y \;\&\; G \models Y \to\to Z \Rightarrow F \bigcup G \models X \to\to Z \setminus Y$;
$F \bigcup G \models X \to\to Y \;\&\; F \bigcup G \models Y \to\to Z \Rightarrow F \bigcup G \models X \to\to Z \setminus Y$;
6) $F \models X \to Y \Rightarrow F \bigcup G \models X \to\to Y$;
$F \bigcup G \models X \to Y \Rightarrow F \bigcup G \models X \to\to Y$;
7) $F \bigcup G \models X \to\to Z \;\&\; F \bigcup G \models Y \to Z' \;\&\; Z' \subseteq Z \;\&\; Y \bigcap Z = \varnothing \Rightarrow F \bigcup G \models$ $X \to Z'$. □

FD or MVD φ is syntactically derived from the set of dependencies ($F \bigcup G \mid -_R \varphi$), if there is a finite sequence of FD or MVD $\varphi_1, \varphi_2, ..., \varphi_{m-1}, \varphi_m$ where $\varphi_m = \varphi$ and for all $\forall i = \overline{1, m-1}$ each φ_i is either the axiom of reflexivity (FD's or MVD's) or belongs to $F \bigcup G$ or is derived with some inference rule (complementation for MVD's, augmentation (for FD's or MVD's), transitivity (for FD's or MVD's), mixed inference rules for FD's and MVD's) from the previous in this sequence φ_j, φ_k, $j, k < i$.

As has been started above, let sequence $\varphi_1, \varphi_2, ..., \varphi_{m-1}, \varphi_m$ be called proof of φ from set of dependencies $F \bigcup G$.

Let there be given certain sets F and G of FD's and MVD's respectively. *Closure* $[F \bigcup G]_R$ – is a set of all FD's and MVD's that are syntactically derived from $F \bigcup G$:

$$[F \bigcup G]_R \overset{def}{=} \{\varphi \mid F \bigcup G \mid - \varphi\}.$$

Lemma 6. Next properties are valid:

1) $F \bigcup G \subseteq [F \bigcup G]$ (*increase*);
2) $[[F \bigcup G]] = [F \bigcup G]$ (*idempotency*);
3) $F' \bigcup G' \subseteq F \bigcup G \Rightarrow [F' \bigcup G'] \subseteq [F \bigcup G]$[4] (*monotonicity*);
4) $[F] \subseteq [F \bigcup G]$, $[G] \subseteq [F \bigcup G]$;
5) $[F] \bigcup [G] \subseteq [F \bigcup G]$. □

From the propositions 1-3 it follows that operator $F \bigcup G \mapsto [F \bigcup G]_R$ is closure operator.

To be mentioned one more mixed rule for FD's and MVD's [5]:

$$\{X \rightarrow\rightarrow Y, X \bigcup Y \rightarrow Z\} \vdash X \rightarrow Z \setminus Y .$$

Closure $[X]_{F \bigcup G,R}$ of a set X (with respect to the set of dependencies $F \bigcup G$ and scheme R) is the family of all right parts MVD's which are syntactically derived from the set $F \bigcup G$:

$$[X]_{F \bigcup G,R} \stackrel{def}{=} \{Y \mid X \rightarrow\rightarrow Y \in [F \bigcup G]_R\} .$$

Obviously, $[X]_{F \bigcup G,R} \neq \varnothing$ since, for example, $X \in [X]_{F \bigcup G,R}$, ($X \rightarrow\rightarrow X$, $X \rightarrow X$ are axioms of reflexivity); the latter statement can be strengthened: actually performed inclusion $2^X \subseteq [X]_{F \bigcup G,R}$, where 2^X – Boolean of a set X .

Let $[X]_F$ – closure of a set X with respect to the set of FD's F [9]. Note that by definition $[X]_F \subseteq R$.

Lemma 7. Next properties are valid:

1) $Y \subseteq [X]_F \Rightarrow Y \in [X]_{F \bigcup G,R}$;
2) $[X]_{F \bigcup G,R} = [[X]_F]_{F \bigcup G,R}$. □

Observe that operator $X \mapsto [X]_{F \bigcup G,R}$ is not closure operator; it is based on the fact that this operator has no idempotency property (notion $[[X]_{F \bigcup G}]_{F \bigcup G,R}$ has no sense).

Basis $[X]^{bas}_{F \bigcup G,R}$ of a set X with respect to the set of dependencies $F \bigcup G$ and scheme R is subset of closure $[X]_{F \bigcup G,R}$, such that:

1) $\forall W (W \in [X]^{bas}_{F \bigcup G,R} \Rightarrow W \neq \varnothing)$ (i.e., basis contains only nonempty sets of attributes);
2) $\forall W_i W_j (W_i W_j \in [X]^{bas}_{F \bigcup G,R} \ \& \ W_i \neq W_j \Rightarrow W_i \bigcap W_j = \varnothing)$ (i.e., sets of basis are pairwise disjoint);

[4] From the fact that sets FD's and MVD's are disjoint it follows that inclusion $F' \bigcup G' \subseteq F \bigcup G$ is equivalent to the conjunction of inclusions $F' \subseteq F$, $G' \subseteq G$.

3) $\forall Y(Y \in [X]_{F\bigcup G,R} \Rightarrow \exists\Im(\Im \subseteq [X]^{bas}_{F\bigcup G,R}\ \&\ \Im$ – finite $\&\ Y = \bigcup_{W\in\Im} W$) (i.e., each set of attributes from closure $[X]_{F\bigcup G,R}$ is equal to finite union of some sets from basis).

Lemma 8. Next properties are valid:

1) $\bigcup_{W\in[X]^{bas}_{F\bigcup G,R}} W = R$ for $X \subseteq R$ (i.e. basic is partition R);

2) $A \in [X]_F \Rightarrow \{A\} \in [X]^{bas}_{F\bigcup G,R}$. □

These lemmas are needed to establish the following main results.

4 Correctness and Completeness of Axiomatic for FD's and MFD's

Let φ – FD or MVD.

Statement 1 (Correctness of axiomatic for FD's and MFD's). If dependency φ is syntactically derived from the set of dependencies $F\bigcup G$, then φ is derived semantically from $F\bigcup G$:

$$F\bigcup G \vdash \varphi \Rightarrow F\bigcup G \models \varphi .$$ □

The proof is carried out by induction in the length of proving.

Statement 2 (Completeness of axiomatic for FD's and MFD's). If dependency φ is derived semantically from the set of dependencies $F\bigcup G$, then φ is syntactically derived from $F\bigcup G$ under the assumption $|R| \geq 2$ and $|D| \geq 2$ [5]:

$$F\bigcup G \models \varphi \Rightarrow F\bigcup G \vdash \varphi .$$ □

Condition $|R| \geq 2$ and $|D| \geq 2$ is obtained through a detailed analysis of the proofs.

Theorem 1. The relations of semantic and syntactic succession coincide for axiomatic of FD's and MFD's under the assumption $|D| \geq 2$ and $|R| \geq 2$:

$$F\bigcup G \models \varphi \Leftrightarrow F\bigcup G \vdash \varphi .$$ □

The proof follows directly from statements 1 and 2.

Analogous theorem holds for axiomatic of MFD's (for axiomatic of FD's see [9]).

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[5] For details see further.

5 Completeness Criteria for Axiomatic of FD's and MFD's

Analysis of the proof of the main result (Theorem 1) shows that it is constructed under the assumption $|D| \geq 2$ and $|R| \geq 2$.

The dependence of coincidence of relations of syntactic and semantic succession for different values of cardinalities of the sets R and D is indicated respectively in the tables 1 and 2. The symbol "+" (respectively "-") in the cell means that these relations coincide (do not coincide respectively) under specified assumptions.

Table 1. All variants of cardinalities of the sets R and D for axiomatic of MFD's

D \ R	|*R*|=0	|*R*|=1	|*R*|≥2
|*D*|=0	+	+	–
|*D*|=1	+	+	–
|*D*|≥2	+	+	+

Table 2. All variants of cardinalities of the sets R and D for axiomatic of FD's and MFD's

D \ R	|*R*|=0	|*R*|=1	|*R*|≥2
|*D*|=0	+	–	–
|*D*|=1	+	–	–
|*D*|≥2	+	+	+

The above table shows the following main results.

Theorem 2. The relations of semantic and syntactic succession coincide for axiomatic MVD's if and only if $|R| \leq 1$ or ($|R| \geq 2$ & $|D| \geq 2$). □

Theorem 3. The relations of semantic and syntactic succession coincide for axiomatic of FD's and MVD's if and only if $|D| \geq 2$ or $|R| = 0$. □

6 Conclusion

In this paper we construct a fragment of the mathematical theory of normalization in relational (table) databases – considered axiomatic for multivalued dependencies and axiomatic for functional and multivalued dependencies. For each axiomatic relations of syntactic and semantic succession are considered and the conditions under which these relations coincide (do not coincide) found.

In particular, it is shown that known in the literature proof of the completeness of these axiomatics constructed under the assumption for scheme R and universal domain D: $|R| \geq 2$, $|D| \geq 2$.

The authors believe that the demonstrated approach and developed mathematical apparatus can be successfully used for other tasks of data modelling.

The next challenge – research of independence of axiomatic' components (axioms and inference rules).

References

1. Bui, D.B., Puzikova, A.V.: Theory of Normalization in Relational Databases (A Survey), http://www.khai.edu/csp/nauchportal/Arhiv/REKS/2014/REKS514/BuyPuz.pdf
2. Darwen, H., Date, C., Fagin, R.: A Normal Form for Preventing Redundant Tuples in Relational Databases. In: Proceedings of the 15th International Conference on Database Theory – ICDT 2012, Berlin, Germany, March 26-30, pp. 114–126 (2012)
3. Bahmani, A., Naghibzadeh, M., Bahmani, B.: Automatic database normalization and primary key generation. In: CCECE/CCGEI, Niagara Falls, Canada, May 5-7, pp. 11–16 (2008)
4. Redko, V.N., et al.: Reliatsiini bazy danykh: tablychni alhebry ta SQL-podibni movy. Akademperiodyka, Kyiv (2001)
5. Beeri, C., Fagin, R., Howard, J.: A complete axiomatization for functional and multivalued dependencies. In: Proceedings of the ACM-SIGMOD Conference, Toronto, Canada, August 3-5, pp. 47–61 (1977)
6. Lyndon, R.: Notes on Logic. D. Van Nostrand Company, Inc., Princeton (1966)
7. Skornyakov, L.A.: Elementyi teorii struktur. Nauka, Moskva (1982)
8. Shoenfield, J.: Mathematical logic. Addison-Wesley (1967)
9. Bui, D.B., Puzikova, A.V.: Completeness of Armstrong's axiomatic. Bulletin of Taras Shevchenko National University of Kyiv. Series Physics & Mathematics 3, 103–108 (2011)