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Axiomatics for multivalued dependencies in table databases: correctness and completeness

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Abstract

Axiomatics for multivalued dependencies in table databases and axiomatics for functional and multivalued dependencies are reviewed. For each axiomatic relations of syntactic and semantic succession are considered. A rigorous and convincing proof of correctness and completeness of these axiomatics (within the paradigm of mathematical logic) is established. In particular, the properties of closures of sets of specified dependencies are investigated. The properties of set-theoretic function restriction have been used as mathematical framework.

Keywords: table databases, functional dependencies, multivalued dependencies, completeness of axiomatic system.

1 Introduction

In spite of the accumulated theoretical researches normalization theory is fragmented and is far from satisfactory conclusion. The works devoted to the ways of solving existing problems of designing database schemas (see, for example, [1]) and improvement of algorithmic systems for normalization (see, for example, [2]) evidence this fact. The process of normalization is based upon functional and multivalued dependencies theory the foundation of which is made by corresponding axiomatics and their completeness. The overview of research sources has shown that these axiomatics lack the proof of completeness that will comply with mathematical rigor.

2 Axiomatic for Multivalued Dependencies

All undefined concepts and notations are used in understanding of monograph [3], in particular, $s|X$ — restriction of the row s to the set of attributes X.

Let t — a table, R — the scheme of the table t (finite set of attributes); X, Y, W, Z — subsets of scheme R; s, s_1, s_2 — the rows of table t. Henceforth we shall assume that set R and *universal domain* D (the set, from which attributes take on values in interpretations) are fixed.

A multivalued dependence (MVD) $X \to\to Y$ is valid on the table t of the scheme R (see, for example, [3]), if for two arbitrary rows s_1, s_2 of table t which coincide on the set of attributes X, there exists row $s_3 \in t$ which is equal to the union of restrictions of the rows s_1, s_2, to the sets of attributes $X \cup Y$ and $R \setminus (X \cup Y)$ respectively:

$$(X \to\to Y)(t) = true \overset{def}{\Leftrightarrow} \forall s_1, s_2 \in t(s_1|X = s_2|X \Rightarrow \exists s_3 \in t(s_3 = \\ = s_1|(X \cup Y) \cup s_2|R \setminus (X \cup Y))).$$

Structure of table t, which complies with MVD $X \to\to Y$, can be represented using the following relation. We say that rows s_1, s_2 of table t are in the relation $=_X$, if they coincide on the set of attributes X:

$$s_1 =_X s_2 \overset{def}{\Leftrightarrow} s_1|X = s_2|X.$$

It is obvious that relation $=_X$ is equivalence relation and therefore it partitions the table t into equivalence classes, which are as follows:

$$[s]_{=_X} = \{s|X\} \bigotimes \pi_Y([s]_{=_X}) \bigotimes \pi_{R\setminus(X\cup Y)}([s]_{=_X}),$$

where s — arbitrary representative of the class.

A table $t(R)$ is the model of a set of MVD's G, if each MVD $X \to\to Y \in G$ is valid on table $t(R)$:

$$t(R) \textit{ is the model of } G \overset{def}{\Leftrightarrow}$$
$$\overset{def}{\Leftrightarrow} \forall(X \to\to Y)(X \to\to Y \in G \Rightarrow (X \to\to Y)(t) = true).$$

The following axioms and inference rules are valid [4].

Axiom of *reflexivity*: $\forall t(X \to\to Y)(t) = true$, where $Y \subseteq X$.

Axiom: $\forall t(X \to\to Y)(t) = true$, where $X \cup Y = R$.

Rule of *complementation*: $(X \to\to Y)(t) = true \Rightarrow (X \to\to R \setminus (X \cup$

$Y))(t) = true$.
Rule of *augmentation*: $(X \to\to Y)(t) = true \,\&\, Z \subseteq W \Rightarrow (X \cup W \to\to Y \cup Z)(t) = true$.
Rule of *transitivity*: $(X \to\to Y)(t) = true \,\&\, (Y \to\to Z)(t) = true \Rightarrow (X \to\to Z \setminus Y)(t) = true$.

The proof in terms of the monograph [3], for example, the axiom of reflexivity is given in [5].

A MVD $X \to\to Y$ is *semantically* deduced from the set of MVD's G, if at each table $t(R)$, which is the model of set G, MVD $X \to\to Y$ is valid too:

$$G \models X \to\to Y \overset{def}{\Leftrightarrow} \forall t(R)(t \text{ is the model of the } G \Rightarrow (X \to\to Y)(t) = true).$$

The relation $\models$ will be called semantic consequence relation.

From above-mentioned axioms and inference rules follow corollaries.

Lemma 1. *The following properties of the semantic consequence relation are valid:*

1. $\emptyset \models X \to\to Y$ *for* $Y \subseteq X$.
2. $\emptyset \models X \to\to Y$ *for* $X \cup Y = R$.
3. $G \models X \to\to Y \Rightarrow G \models X \to\to R \setminus (X \cup Y)$.
4. $G \models X \to\to Y \,\&\, Z \subseteq W \Rightarrow G \models X \cup W \to\to Y \cup Z$.
5. $G \models X \to\to Y \,\&\, G \models Y \to\to Z \Rightarrow G \models X \to\to Z \setminus Y$.
6. $G \models X \to\to Y \,\&\, G \models Y \to\to Z \,\&\, Z \cap Y = \emptyset \Rightarrow G \models X \to\to Z$.

A MVD $X \to\to Y$ is *syntactically* derived from the set of MVD's G with respect to the scheme R ($G \vdash_R X \to\to Y$), if there is a finite sequence of MVD's $\varphi_1, \varphi_2, \ldots, \varphi_{m-1}, \varphi_m$, where $\varphi_m = X \to\to Y$ and for all $\forall i = \overline{1, m-1}$ each φ_i is either the axiom of reflexive or belongs to G, or is derived with some inference rule for MVD's (complementation, augmentation, transitivity) from the previous in this sequence φ_j, φ_k, $j, k < i$.

Let sequence $\varphi_1, \varphi_2, \ldots, \varphi_{m-1}, \varphi_m$ be called proof, following the tradition of mathematical logic [6].

Let there be given certain set of MVD's G. *Closure* $[G]_R$ is a set of all MVD's, that are syntactically derived from G:

$$[G]_R \overset{def}{=} \{X \to\to Y | G \vdash_R X \to\to Y\}.$$

For notational convenience, we write $\vdash$ for $\vdash_R$.

Lemma 2. *The following properties are valid:*
1) $G \subseteq [G]$ *(increase);*
2) $[[G]] = [G]$ *(idempotency);*
3) $G \subseteq H \Rightarrow [G] \subseteq [H]$ *(monotonicity).*

The proofs of this properties are given in [5].
Thereby, operator $G \mapsto [G]$ is closure operator in terms of [7].
From reflexivity axiom and inference rules indicated above it is possible to get other inference rules for MVD's [4].
Rule of *pseudo-transitivity*:
$\{X \rightarrow\rightarrow Y, Y \cup W \rightarrow\rightarrow Z\} \vdash X \cup W \rightarrow\rightarrow Z \setminus (Y \cup W)$.
Rules of *difference*:
1. $\{X \rightarrow\rightarrow Y\} \vdash X \rightarrow\rightarrow Y \setminus X$;
2. $\{X \rightarrow\rightarrow Y \setminus X\} \vdash X \rightarrow\rightarrow Y$;
3. $\{X \rightarrow\rightarrow Y\} \vdash X \rightarrow\rightarrow R \setminus Y$.

Rule of *union*: $\{X \rightarrow\rightarrow Y_1, X \rightarrow\rightarrow Y_2\} \vdash X \rightarrow\rightarrow Y_1 \cup Y_2$.
Rules of *decomposition*:
1. $\{X \rightarrow\rightarrow Y_1, X \rightarrow\rightarrow Y_2\} \vdash X \rightarrow\rightarrow Y_1 \cap Y_2$;
2. $\{X \rightarrow\rightarrow Y_1, X \rightarrow\rightarrow Y_2\} \vdash X \rightarrow\rightarrow Y_1 \setminus Y_2$.

Lemma 3. *The following properties are valid for* $n = 2, 3, \ldots$*:*
1. $\{X \rightarrow\rightarrow Y_1, \ldots, X \rightarrow\rightarrow Y_n\} \vdash X \rightarrow\rightarrow Y_1 \cup \ldots \cup Y_n$*;*
2. $\{X \rightarrow\rightarrow Y_1, \ldots, X \rightarrow\rightarrow Y_n\} \vdash X \rightarrow\rightarrow Y_1 \cap \ldots \cap Y_n$.

The proof of this lemma is constructed by the induction in the n, according to the rules of augmentation and transitivity.

3 Axiomatic for FD's and MFD's

It will be recalled that a functional dependence $X \rightarrow Y$ is valid on the table t, if for two arbitrary rows s_1, s_2 of table t which coincide on the set of attributes X, their equality on the set of attributes Y is fulfilled (see, for example [3]), that is:

$$(X \rightarrow Y)(t) = true \overset{def}{\Leftrightarrow} \forall s_1, s_2 \in t(s_1|X = s_2|X \Rightarrow s_1|Y = s_2|Y).$$

Let there be given sets F and G of FD's and MVD's respectively. A table $t(R)$ is the *model* of a set $F \cup G$, if each dependency $\varphi \in F \cup G$ is valid on table t:

$$t(R) \textit{ is model of } F \cup G \overset{def}{\Leftrightarrow} \forall \varphi(\varphi \in F \cup G \Rightarrow \varphi(t) = true).$$

Mixed inference rules for FD's and MVD's are valid [4].

1. Rule of *extension* FD to MVD: $(X \to Y)(t) = true \Rightarrow (X \to\to Y)(t) = true$.

2. $(X \to\to Z)(t) = true \,\&\, (Y \to Z')(t) = true \,\&\, Z' \subseteq Z \,\&\, Y \cap Z = \emptyset \Rightarrow (X \to Z')(t) = true$.

The proof of extension rule is given, for example, in the monograph [3, p. 73] but the proof of rule 2 — in [5].

FD or MVD φ is *semantically* deduced from the set of dependencies $F \cup G$, if at each table $t(R)$, which is the model of a set of dependencies $F \cup G$, dependency φ is valid too:

$$F \cup G \models \varphi \overset{def}{\Leftrightarrow} \forall t(R)(t \text{ model of } F \cup G \Rightarrow \varphi(t) = true).$$

From above-mentioned mixed inference rules for FD's and MVD's follow corollaries (the properties of semantic consequence relation):

1. $F \models X \to Y \Rightarrow F \models X \to\to Y$;

2. $G \models X \to\to Z \,\&\, F \models Y \to Z' \,\&\, Z' \subseteq Z \,\&\, Y \cap Z = \emptyset \Rightarrow F \cup G \models X \to Z'$.

Lemma 4. *Let H_1 and H_2 — the sets of dependencies (FD's or MVD's) and T_1, T_2 — the sets of all their models respectively. Then implication $H_1 \subseteq H_2 \Rightarrow T_1 \supseteq T_2$ is carried out.*

Corollary 1. *The following properties of the semantic consequence relation are valid:*

1. $F \models \varphi \Rightarrow F \cup G \models \varphi$;

2. $G \models \varphi \Rightarrow F \cup G \models \varphi$.

Lemma 5. *The following properties of the semantic consequence relation are valid:*

1) $F \models X \to Y \Rightarrow F \cup G \models X \cup Z \to Y \cup Z$ *for* $Z \subseteq R$;

$F \cup G \models X \to Y \Rightarrow F \cup G \models X \cup Z \to Y \cup Z$ *for* $Z \subseteq R$;

2) $F \models X \to Y \& F \models Y \to Z \Rightarrow F \cup G \models X \to Z$;

$F \cup G \models X \to Y \& F \cup G \models Y \to Z \Rightarrow F \cup G \models X \to Z$;

3) $G \models X \to\to Y \Rightarrow F \cup G \models X \to\to R \setminus (X \cup Y)$;
$F \cup G \models X \to\to Y \Rightarrow F \cup G \models X \to\to R \setminus (X \cup Y)$;
4) $G \models X \to\to Y \& Z \subseteq W \Rightarrow F \cup G \models X \cup W \to\to Y \cup Z$;
$F \cup G \models X \to\to Y \& Z \subseteq W \Rightarrow F \cup G \models X \cup W \to\to Y \cup Z$;
5) $G \models X \to\to Y \& G \models Y \to\to Z \Rightarrow F \cup G \models X \to\to Z \setminus Y$;
$F \cup G \models X \to\to Y \& F \cup G \models Y \to\to Z \Rightarrow F \cup G \models X \to\to Z \setminus Y$;
6) $F \models X \to Y \Rightarrow F \cup G \models X \to\to Y$;
$F \cup G \models X \to Y \Rightarrow F \cup G \models X \to\to Y$;
7) $F \cup G \models X \to\to Z \& F \cup G \models Y \to Z' \& Z' \subseteq Z \& Y \cap Z = \emptyset \Rightarrow F \cup G \models X \to Z'$.

FD or MVD φ is *syntactically* derived from the set of dependencies $F \cup G$ ($F \cup G \vdash_R \varphi$), if there is a finite sequence of FD or MVD $\varphi_1, \varphi_2, \ldots, \varphi_{m-1}, \varphi_m$, where $\varphi_m = \varphi$ and for all $\forall i = \overline{1, m-1}$ each φ_i is either the axiom of reflexivity (FD's or MVD's) or belongs to $F \cup G$ or is derived with some inference rule (complementation for MVD's, augmentation (for FD's or MVD's), transitivity (for FD's or MVD's), mixed inference rules for FD's and MVD's) from the previous in this sequence φ_j, φ_k, $j, k < i$.

As it has been stated above, let sequence $\varphi_1, \varphi_2, \ldots, \varphi_{m-1}, \varphi_m$ be called proof of φ from set of dependencies $F \cup G$.

Let there be given certain sets F and G of FD's and MVD's respectively.

Closure $[F \cup G]_R$ — is a set of all FD's and MVD's that are syntactically derived from $F \cup G$: $[F \cup G]_R \stackrel{def}{=} \{\varphi | F \cup G \vdash_R \varphi\}$.

Lemma 6. *The following properties are valid:*
1) $F \cup G \subseteq [F \cup G]$ *(increase);*
2) $[[F \cup G]] = [F \cup G]$ *(idempotency);*
3) $F' \cup G' \subseteq F \cup G \Rightarrow [F' \cup G'] \subseteq [F \cup G]$ *(monotonicity).*
4) $[F] \subseteq [F \cup G]$, $[G] \subseteq [F \cup G]$, $[F] \cup [G] \subseteq [F \cup G]$.

From the propositions 1-3 it follows that operator $F \cup G \mapsto [F \cup G]_R$ is the closure operator.

Closure $[X]_{F \cup G, R}$ of a set X (with respect to the set of dependencies $F \cup G$ and scheme R) is the family of all right parts of MVD's which are syntactically derived from the set $F \cup G$:

$$[X]_{F\cup G,R} \stackrel{def}{=} \{Y | X \to\to Y \in [F \cup G]_R\}.$$

Obviously, $[X]_{F\cup G,R} \neq \emptyset$ since, for example, $X \in [X]_{F\cup G,R}$, ($X \to\to X$, $X \to X$ are axioms of reflexivity); the latter statement can be strengthened: actually performed inclusion $2^X \subseteq [X]_{F\cup G,R}$, where 2^X — Boolean of a set X.

Let $[X]_F$ — closure of a set X with respect to the set of FD's F [8]. Note that by definition $[X]_F \subseteq R$.

Lemma 7. *The following properties are valid:*

1. $Y \subseteq [X]_F \Rightarrow Y \in [X]_{F\cup G,R}$;

2. $[X]_{F\cup G,R} = [[X]_F]_{F\cup G,R}$.

Proof. To prove proposition 1 we will construct a proof of MVD $X \to\to Y$ from set of dependences $F \cup G$. Really, we have:

1. Proof of FD $X \to [X]_F$ from F ([8], lemma 9);
2. $[X]_F \to Y$ (axiom of reflexivity for FD's; by assumption, $Y \subseteq [X]_F$);
3. $X \to Y$ (with 1 and 2 according to the rule of transitivity for FD);
4. $X \to\to Y$ (with 3 according to the rule of extension FD to MVD).

Thus, by definition of closure $[X]_{F\cup G,R}$ it follows $Y \in [X]_{F\cup G,R}$.

Let's prove proposition 2. Let $Y \in [X]_{F\cup G,R}$; let's show that $Y \in [[X]_F]_{F\cup G,R}$. By definition of closure $[X]_{F\cup G,R}$ there is proof of MVD $X \to\to Y$ from the set of dependencies $F \cup G$. Let's make a proof of MVD $[X]_F \to\to Y$ from $F \cup G$.

1. $[X]_F \to\to X$ (axiom of reflexivity for MVD's because $X \subseteq [X]_F$ according to [8, lemma 9]);
2. Proof of MVD $X \to\to Y$ from $F \cup G$ which exists by assumption;
3. $[X]_F \to\to Y \setminus X$ (with 1 and 2 according to the rule of transitivity for MVD's);
4. $[X]_F \to\to Y$ (with 3 according to the rule of augmentation for MVD's which can be obtained by simplification of the MVD $[X]_F \cup (X \cap Y) \to\to Y \setminus X \cup (X \cap Y)$; really, $Y \setminus X \cup (X \cap Y) = Y$; $[X]_F \cup (X \cap Y) = [X]_F$, because $X \cap Y \subseteq [X]_F$).

Thus, we have $Y \in [[X]_F]_{F\cup G,R}$.

Let now $Y \in [[X]_F]_{F\cup G,R}$; let's show that $Y \in [X]_{F\cup G,R}$. By definition of closure $[[X]_F]_{F\cup G,R}$ there is proof of MVD $[X]_F \to\to Y$

from the set of dependencies $F \cup G$. Let's make a proof of MVD $X \to\to Y$ from $F \cup G$.
1. Proof of FD $X \to [X]_F$ from F ([8], lemma 9);
2. $X \to\to [X]_F$ (with 1 according to the rule of extension FD to MVD);
3. Proof of MVD $[X]_F \to\to Y$ from set $F \cup G$ which exists by assumption;
4. $X \to\to Y \setminus [X]_F$ (from the latest MVD's in sequences of proof of items 2 and 3 according to the rule of transitivity for MVD's);
5. $[X]_F \to [X]_F \cap Y$ (axiom of reflexivity for FD's);
6. $X \to [X]_F \cap Y$ (with 1 and 5 according to the rule of transitivity for FD's);
7. $X \to\to [X]_F \cap Y$ (with 6 according to the rule of extension FD to MVD);
8. $X \to\to Y$ (with 4 and 7 according to the additional rule for MVD's we have MVD $X \to\to (Y \setminus [X]_F) \cup ([X]_F \cap Y)$; which has to be simplified).

Thus, $Y \in [X]_{F\cup G,R}$.$\square$

Observe that operator $X \mapsto [X]_{F\cup G,R}$ is not closure operator; it is based on the fact that this operator has no idempotency property (notion $[[X]_{F\cup G,R}]_{F\cup G,R}$ has no sense).

Basis $[X]^{bas}_{F\cup G,R}$ of a set X with respect to the set of dependencies $F \cup G$ and scheme R is subset of closure $[X]_{F\cup G,R}$, such that:

1. $\forall W (W \in [X]^{bas}_{F\cup G,R} \Rightarrow W \neq \emptyset$ (i.e., basis contains only nonempty sets of attributes);

2. $\forall W_i, W_j (W_i, W_j \in [X]^{bas}_{F\cup G,R} \,\&\, W_i \neq W_j \Rightarrow W_i \cap W_j = \emptyset)$ (i.e., sets of basis are pairwise disjoint);

3. $\forall Y (Y \in [X]_{F\cup G,R} \Rightarrow \exists \mathcal{T} (\mathcal{T} \subseteq [X]^{bas}_{F\cup G,R} \,\&\, \mathcal{T} - finite \,\&\, Y = \bigcup_{W\in\mathcal{T}} W)$ (i.e., each set of attributes from closure $[X]_{F\cup G,R}$ is equal to finite union of some sets from basis).

Lemma 8. *The following properties are valid:*

1. $\bigcup_{W\in[X]^{bas}_{F\cup G,R}} W = R$ *for* $X \subseteq R$ *(i.e. basic is partition of* R*);*

2. $A \in [X]_F \Rightarrow \{A\} \in [X]^{bas}_{F\cup G,R}$.

These lemmas are needed to establish the following main results.

4 Correctness and Completeness of Axiomatic for FD's and MVD's

Let φ — FD or MVD.

Statement 1 (Correctness of axiomatic for FD's and MVD's). *If dependency φ is syntactically derived from the set of dependencies $F \cup G$, then φ is derived semantically from $F \cup G$:*

$$F \cup G \vdash \varphi \Rightarrow F \cup G \models \varphi.$$

Proof. The proof is carried out by induction in the course of proving.

Basis. The length of proof is equal to 1. It means that dependency φ is either trivial (FD or MVD) or $\varphi \in F \cup G$. In all these cases (if φ is the trivial FD, then use corollary 1 from [8]; if φ is the trivial MVD, then use lemma 1, proposition 1) semantic succession $F \cup G \models \varphi$ takes place.

Inductive step. Let $\varphi_1, \varphi_2, \ldots, \varphi_{m-1}, \varphi_m$, $m \geq 2$ is a proof of dependency φ (FD or MVD) from set $F \cup G$. Let us consider all possible cases for last element of sequence φ_m, where φ_m — FD or MVD.

The case when φ_m is either trivial (FD or MVD) or $\varphi_m \in F \cup G$ we consider in a way analogous to that used in the basis of induction.

Let φ_m — FD which is deduced from certain FD φ_i, $i < m$ according to the rule of completion. It is obvious that $F \cup G \vdash \varphi_i$; by induction assumption we have $F \cup G \models \varphi_i$. It remains to use lemma 5, proposition 1.

Similar cases are considered, where:

– φ_m — FD that results from previous in this sequence of FD's according to the rule of transitivity (lemma 5, proposition 2 are used);

– φ_m — MVD that results from MVD φ_i, where $i < m$, according to the rules of complementation or augmentation (lemma 5, proposition 3 for the rule of complementation or lemma 5, proposition 4 for the rule of augmentation are used);

– φ_m — MVD that results from previous in this sequence of MVD's according to the rule of transitivity (lemma 5, proposition 5 are used);

– φ_m — MVD that results from FD φ_i, where $i < m$, according to the rule of extension of FD to MVD (lemma 5, proposition 6 are used);

– φ_m — FD that results from previous in this sequence of MVD and FD according to the mixed inference rule for FD's and MVD's (lemma 5, proposition 7 are used).$\square$

Statement 2 (Completeness of axiomatic for FD's and MVD's). *If dependency φ is derived semantically from the set of dependencies $F \cup G$, then φ is syntactically derived from $F \cup G$ under the assumption $|R| \geq 2$ and $|D| \geq 2$:*

$$F \cup G \models \varphi \Rightarrow F \cup G \vdash \varphi.$$

Proof. We now turn to the idea of proof [4], which we reconstruct and complement. We will prove our statement by contradiction. Let the set $F \cup G$ and the dependency φ (FD or MVD) are such that $F \cup G \models \varphi$ is fulfilled, but $F \cup G \vdash \varphi$ is not valid, that is $\varphi \notin [F \cup G]$.

To show the contradiction with $F \cup G \models \varphi$, make such model of set $F \cup G$ that dependency φ is not valid.

Let us fix two distinct elements a and b in the universal domain. Let the set X is the left part of dependency φ (FD or MVD). Let the cover $[X]_F$ consists of attributes $A_1, A_2, \dots, A_k$. According to property 2, lemma 8 for $i = \overline{1,k}$ we have $\{A_i\} \in [X]^{bas}_{F \cup G,R}$, that is basis $[X]^{bas}_{F \cup G,R}$ partitions the scheme R of cardinality n on the sets $\{A_1\} = W_1, \{A_2\} = W_2, \dots, \{A_k\} = W_k, W_{k+1}, \dots, W_m$, where $m \leq n$.

The table t is constructed as follows: the number of rows is 2^{m-k}; for all attributes $A \in [X]_F$ each row $s \in t$ takes values only from the set $\{a\}$, that is $s(A) = a$; at the sets W_i for $i = \overline{k+1, m}$, rows take values either only from the set $\{a\}$ or only from the set $\{b\}$ (see Table 1).

Table 1. Table t from the proof of Statement 2

$[X]_F$	W_{k+1}	W_{k+2}	$\dots$	W_{m-1}	W_m
a	a	a	$\dots$	a	a
a	a	a	$\dots$	a	b
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
a	b	b	$\dots$	b	b

Consider two possible cases for φ.

Let φ — FD of the form $X \to Y$ such that $X \to Y \notin [F \cup G]$. Let's show that FD $X \to Y$ is not valid on the table t. Since $X \to Y \notin [F \cup G]$, then from inclusion $[F] \subseteq [F \cup G]$ (lemma 6, property 4) it follows $X \to Y \notin [F]$. Hence we have $Y \not\subseteq [X]_F \subset R$ ([8], lemma 9, property 2), that is $Y \cap R \setminus [X]_F \neq \emptyset$. Therefore, for arbitrary attribute $A \in Y \cap R \setminus [X]_F$ there exist such rows s_1 and s_2, that $s_1(A) = a$ and $s_2(A) = b$ (by construction of table t), hence, $s_1|Y \neq s_2|Y$. On account of the equality $s_1|X = s_2|X$ (by construction of table t) we have $(X \to Y)(t) = false$.

Let φ — MVD of the form $X \to\to Y$ such that $X \to\to Y \notin [F \cup G]$. Let's show that MVD $X \to\to Y$ is not valid on the table t. By assumption $X \to\to Y \notin [F \cup G]$, it follows:

1. $Y \not\subseteq [X]_F$, because then the MVD $X \to\to Y$ will have a proof (and hence will belong to the set $[F \cup G]$):

a. Proof of FD $X \to [X]_F$ from the set F, and therefore from the set $F \cup G$ ([8], lemma 9);

b. $[X]_F \to Y$ (axiom of reflexivity for FD's; recall that by assumption $Y \subseteq [X]_F$);

c. $X \to Y$ (with a and b according to the rule of transitivity for FD's);

d. $X \to\to Y$ (with c according to the rule of extension FD to MVD);

2. $Y \neq \bigcup W_i$ for some $1 \leq i \leq m$ because $W_i \in [X]^{bas}_{F \cup G, R}$, that is MVD $X \to\to \bigcup W_i \in [F \cup G]$;

3. Note also that $Y \neq \emptyset$ because MVD $X \to\to \emptyset$ is the axiom of reflexivity and it belongs to $[F \cup G]$;

4. It still remains to consider the case where $Y \cap W_i \subset W_i$, $Y \cap W_i \neq \emptyset$ for some $k + 1 \leq i \leq m$. Suppose that MVD $X \to\to Y$ holds at the table t. Since for arbitrary rows s_1 and s_2 the equality $s_1|X = s_2|X$ is fulfilled (by construction of table t), then for fixed i we choose s_1 and s_2 as follows: $range(s_1|W_i) = \{a\}$ and $range(s_2|W_i) = \{b\}$. According to the rules of decomposition (item 2) we have $\{X \to\to W_i, X \to\to Y\} \vdash X \to\to W_i \setminus Y$; it follows that there exists row s_3, that $s_3|W_i = s_1|(W_i \setminus Y) \cup s_2|(W_i \cap Y)$, that is $s_3|W_i$ takes values from both sets $\{a\}$ and $\{b\}$; this contradicts the construction of the table t.

Thus, MVD $X \to\to Y$ which does not belong to the set $[F \cup G]$, is not valid on the table t.

Let's now show that table t is the model of set $F \cup G$. Consider two possible cases.

I. Given FD $U \to Z \in F \subseteq F \cup G$. We will show that $(U \to Z)(t) = true$, that is for arbitrary rows s_1 and s_2 the implication $s_1|U = s_2|U \Rightarrow s_1|Z = s_2|Z$ is valid.

There are two possible cases for the set of attributes U.

Case 1: $U \cap R \setminus [X]_F = \emptyset$ that is $U \subseteq [X]_F$. Then for arbitrary rows s_1 and s_2 by construction of the table t we have $s_1|U = s_2|U$; so we need to show the equality $s_1|Z = s_2|Z$. To prove this, it is sufficient to make sure that $Z \subseteq [X]_F$. For this purpose we consider the following proof of FD $X \to Z$ from the set F:
1. Proof of FD $X \to [X]_F$ from the set F ([8], lemma 9);
2. $[X]_F \to U$ (axiom of reflexivity for FD's; recall that by assumption $U \subseteq [X]_F$);
3. $X \to U$ (the rule of transitivity is applied to the FD $X \to [X]_F$, which is the last element of proof 1 and FD $[X]_F \to U$ 2);
4. $U \to Z$ (element of set F);
5. $X \to Z$ (with 3 and 4 according to the rule of transitivity for FD's).

Hence we have $F \vdash X \to Z$, that is $Z \subseteq [X]_F$; it follows that $s_1|Z = s_2|Z$.

Case 2: $U \cap R \setminus [X]_F \neq \emptyset$. In the case when $Z \subseteq [X]_F$, FD $U \to Z$ is valid trivially on account of the construction of the table t.

Let $Z \nsubseteq [X]_F$. We first show that FD $U \to Z$, where $Z \cap W_i \neq \emptyset$ and at that $U \cap W_i = \emptyset$, for $k+1 \leq i \leq m$, does not belong to the set F. Assume the contrary. Then there exists proof for FD, which is not valid on the table t:
1. $U \to Z$ (element of set F);
2. $Z \to Z \cap W_i$ (axiom of reflexivity for FD's);
3. $U \to Z \cap W_i$ (with 1 and 2 according to the rule of transitivity for FD's);
4. $Z \cap W_i \to W_i$ (by construction of the table t; recall that $\forall A', A'' \in W_i(s(A') = s(A'')))$);
5. $U \to W_i$ (with 3 and 4 according to the rule of transitivity for FD's);

6. $X \rightarrow\rightarrow W_i$ (by construction of the table t; recall that $W_i \in [X]^{bas}_{F\cup G,R}$);
7. $X \rightarrow W_i$ (with 6 and 5 according to the mixed inference rule for FD's and MVD's; recall that by assumption $U \cap W_i = \emptyset$).

Thus, for some i, $k+1 \leq i \leq m$ we have the proof of FD $X \rightarrow W_i$, which is not valid on the table t (by construction of the table t). Therefore, FD $U \rightarrow Z$ where $Z \cap W_i \neq \emptyset$ and at that $U \cap W_i = \emptyset$ does not belong to the set F.

Considering that $\bigcup_{W_i \in [X]^{bas}_{F\cup G,R}} W_i = R$ (property 1, lemma 8) we write the set Z in the form $Z = \bigcup_{i=1}^{m}(Z \cap W_i)$. Fix i and show that FD $U \rightarrow Z \cap W_i$ is valid on the table t. Let's consider all possible cases:
1. if $Z \cap W_i \subseteq [X]_F$, then FD $U \rightarrow Z \cap W_i$ is valid on the table t (by construction);
2. if $Z \cap W_i \subseteq W_i$ for $k+1 \leq i \leq m$ then $U \cap W_i \neq \emptyset$ as it has been showed. Let's make the proof of FD $U \rightarrow Z \cap W_i$:

a. $U \rightarrow U \cap W_i$ (axiom of reflexivity for FD's);

b. $U \cap W_i \rightarrow Z \cap W_i$ (by construction of the table t; recall that $\forall A', A'' \in W_i (s(A') = s(A''))$ and on account of the inclusions $U \cap W_i \subseteq W_i$, $Z \cap W_i \subseteq W_i$);

c. $U \rightarrow Z \cap W_i$ (with a and b according to the rule of transitivity for FD's).

Thus, FD $U \rightarrow Z \cap W_i$, $1 \leq i \leq m$, is valid on the table t; hence FD $U \rightarrow \bigcup_{i=1}^{m}(Z \cap W_i)$ is valid [8, lemma 7, conclusion 6], therefore FD $U \rightarrow Z$ is valid.

II. Let's consider MVD $U \rightarrow\rightarrow Z \in G \subseteq F \cup G$ and show that $(U \rightarrow\rightarrow Z)(t) = true$.

We first show that MVD $U \rightarrow\rightarrow Z$, where $Z \cap W_i \subset W_i$ $(Z \cap W_i \neq \emptyset)$ and at that $U \cap W_i = \emptyset$ for $k+1 \leq i \leq m$, does not belong to the set G. Assume the contrary. Then there exists proof for MVD, which is not valid on the table t:
1. $U \rightarrow\rightarrow Z$ (element of set G);
2. $R \backslash W_i \rightarrow\rightarrow Z$ (from inclusion $R \backslash W_i \supseteq Z \backslash W_i$ and with 1 according to the rule of augmentation for MVD's we have $U \cup R \backslash W_i \rightarrow\rightarrow Z \cup Z \backslash W_i$; it remains to consider that $U \cap W_i = \emptyset$);

3. $X \to\to W_i$ (by construction of the table t; recall that $W_i \in [X]^{bas}_{F\cup G,R}$);
4. $X \to\to R \setminus W_i$ (with 3 according to the rule of difference (item 3));
5. $X \to\to Z \cap W_i$ (with 4 and 2 according to the rule of transitivity for MVD's we have $X \to\to Z \setminus (R \setminus W_i)$, which should be simplified).

Considering that $Z \cap W_i \subset W_i$ ($Z \cap W_i \neq \emptyset$) we have contradiction with assumption that W_i belongs to basis $[X]^{bas}_{F\cup G,R}$. Thus, MVD $U \to\to Z$, where $Z \cap W_i \subset W_i$ ($Z \cap W_i \neq \emptyset$) and at that $U \cap W_i = \emptyset$ for $k+1 \leq i \leq m$, does not belong to the set G.

On account of the property $\bigcup_{W_i \in [X]^{bas}_{F\cup G,R}} W_i = R$ (property 1, lemma 8) write the set Z in the form $Z = \bigcup_{i=1}^{m}(Z \cap W_i)$. Fix i and show that MVD $U \to\to Z \cap W_i$ is valid on the table t. Let's consider all possible cases for $Z \cap W_i$:
1. $Z \cap W_i = \emptyset$; then $U \to\to \emptyset$ is an axiom of reflexivity and is valid trivially;
2. $Z \cap W_i = W_i$; then $U \to\to W_i$ is valid by construction of the table t (recall that table t consists of combinations of all possible values on the sets of attributes W_i and $R \setminus W_i$, $k+1 \leq i \leq m$);
3. $Z \cap W_i \subseteq [X]_F$; then $U \to\to Z \cap W_i$ holds true by construction of table t;
4. $Z \cap W_i \subset W_i$ for $k+1 \leq i \leq m$ and $Z \cap W_i \neq \emptyset$, then $U \cap W_i \neq \emptyset$ as it has been showed.

Let's show that $U \to\to Z \cap W_i$ is valid for this case; that is for arbitrary rows s_1 and s_2 such that $s_1|U = s_2|U$, there exists row s_3 that $s_3 = s_1|U \cup s_1|(Z \cap W_i) \cup s_2|R \setminus (U \cup (Z \cap W_i))$. By conditions $s_1|U = s_2|U$ and $U \cap W_i \neq \emptyset$, it follows equality $s_1|W_i = s_2|W_i$ (by construction of the table t). Restrict both parts of this equality to the set Z: $(s_1|W_i)|Z = (s_2|W_i)|Z$. According to the property of restriction operator ($(U|Y)|Z = (U|(Y \cap Z)$ [3, p. 24]) it follows $(s_1|(Z \cap W_i) = s_2|(Z \cap W_i)$. Thus, $s_3 = s_2|U \cup s_2|(Z \cap W_i) \cup s_2|R \setminus (U \cup (Z \cap W_i)) = s_2 \in t$. Consequently, MVD $U \to\to Z \cap W_i$ is valid on table t.

From the above and by lemma 3, item 1 it follows $\{U \to\to Z \cap W_1, \ldots, U \to\to Z \cap W_k\} \vdash U \to\to Z$. Therefore, MVD $U \to\to Z$ is valid on table t.$\square$

Conditions $|R| \geq 2$ and $|D| \geq 2$ are obtained through a detailed analysis of the proofs.

Theorem 1. *The relations of semantic and syntactic succession coincide for axiomatic of FD's and MVD's under the assumption* $|R| \geq 2$ *and* $|D| \geq 2$*:*

$$F \cup G \models \varphi \Leftrightarrow F \cup G \vdash \varphi.$$

The proof follows directly from Statements 1 and 2 (Section 4).

Analogous theorem holds for axiomatic of MVD's (for axiomatic of FD's see [8]).

5 Conclusion

In this paper we have constructed a fragment of the mathematical theory of normalization in table databases — axiomatics for multivalued dependencies and axiomatics for functional and multivalued dependencies are considered. In particular, it was produced the proof of correctness of these axiomatics and completely reconstructed the known in the literature proof of the completeness.

References

[1] H. Darwen, C. Date, R. Fagin. *A Normal Form for Preventing Redundant Tuples in Relational Databases.* 15th International Conference on Database Theory ICDT2012 Proc. Berlin, Germany, March 26–30, 2012, pp. 114–126.

[2] A. Bahmani, M. Naghibzadeh, B. Bahmani. *Automatic database normalization and primary key generation.* CCECE/CCGEI Proc. Niagara Falls, Canada, May 5-7, 2008, pp. 11–16.

[3] V. N. Redko, Yu.Y. Brona, D.B. Bui, S.A. Polyakov. *Relational Data Bases: Table Algebras and SQL-like Languages.* Akademperiodyka, Kyiv, 2001 (in Ukrainian).

[4] C. Beeri, R. Fagin, J. Howard. *A complete axiomatization for functional and multivalued dependencies.* ACM-SIGMOD Conference,Toronto, Canada, Aug. 3–5, 1977, pp. 47–61.

[5] D. Bui, A. Puzikova. *Axiomatics for multivalued dependencies in table databases: correctness, completeness, completeness criteria.* 5th CrISS-DESSERT Workshop,Brunow, Poland, June 29 - July 3, 2015.

[6] R. Lyndon. *Notes on Logic.* Van Nostrand Company, Inc., Princeton, 1966.

[7] L. A. Skornyakov. *Elements of the theory of structures.* Nauka, Moscow, 1982 (in Russian).

[8] D. B. Bui, A. V. Puzikova. *Completeness of Armstrongs axiomatic.* Bulletin of Taras Shevchenko National University of Kyiv. Series Physics and Mathematics, no. 3 (2011), pp. 103–108.

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